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On powers of class $A(k)$ operators including p -hyponormal and log-hyponormal operators

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ABSTRACT

In [15], we introduced class $A(k)$ as a class of operators including p -hyponormal and log-hyponormal operators. In this report, we shall show that “if T is an invertible class $A(k)$ operator for $k \in (0, 1]$, then T^n is a class $A(\frac{k}{n})$ operator for all positive integer n .”

Moreover, we shall show a similar result on powers of class $AI(s, t)$ operators which were introduced in [7] as extensions of class $A(k)$ operators, that is, “if T is a class $AI(s, t)$ operator for $s, t \in (0, 1]$, then T^n is a class $AI(\frac{s}{n}, \frac{t}{n})$ operator for all positive integer n .”

1. INTRODUCTION

In what follows, a capital letter means a bounded linear operator on a complex Hilbert space H . An operator T is said to be positive (denoted by $T \geq 0$) if $(Tx, x) \geq 0$ for all $x \in H$.

An operator T is said to be p -hyponormal if $(T^*T)^p \geq (TT^*)^p$ for a positive number p , and T is said to be log-hyponormal if T is invertible and $\log T^*T \geq \log TT^*$. p -Hyponormal and log-hyponormal operators were defined as extensions of hyponormal operators, i.e., $T^*T \geq TT^*$. It is well known that “every p -hyponormal operator is a q -hyponormal operator for $p \geq q > 0$ ” by the celebrated Löwner-Heinz theorem “ $A \geq B \geq 0$ ensures $A^\alpha \geq B^\alpha$ for any $\alpha \in [0, 1]$,” and “every invertible p -hyponormal operator is a log-hyponormal operator” since $\log t$ is an operator monotone function. It is also well known that there exists a hyponormal operator T such that T^2 is not a hyponormal operator [16, Problem 209]. Related to this fact, the following result on powers of p -hyponormal operators for $p > 0$ was shown by Aluthge-Wang [3] and Ito [19].

Theorem A.1 ([3, 19]). *If T is a p -hyponormal operator for $p > 0$, then T^n is a $\min\{1, \frac{p}{n}\}$ -hyponormal operator for all positive integer n .*

We remark that Aluthge and Wang [3] showed Theorem A.1 in case $p \in (0, 1]$. Then Ito [19] showed Theorem A.1 in case $p > 0$. Moreover, we obtained the following result for log-hyponormal operators.

Theorem A.2 ([23]). *If T is a log-hyponormal operator, then T^n is also a log-hyponormal operator for all positive integer n .*

We remark that the best possibilities of Theorem A.1 and Theorem A.2 were shown in [14, 19]. Theorem A.1 and Theorem A.2 were shown as nice applications of the following Theorem F.

Theorem F (Furuta inequality [9]).

If $A \geq B \geq 0$, then for each $r \geq 0$,

$$(i) \quad (B^{\frac{r}{2}} A^p B^{\frac{r}{2}})^{\frac{1}{q}} \geq (B^{\frac{r}{2}} B^p B^{\frac{r}{2}})^{\frac{1}{q}}$$

and

$$(ii) \quad (A^{\frac{r}{2}} A^p A^{\frac{r}{2}})^{\frac{1}{q}} \geq (A^{\frac{r}{2}} B^p A^{\frac{r}{2}})^{\frac{1}{q}}$$

hold for $p \geq 0$ and $q \geq 1$ with $(1+r)q \geq p+r$.

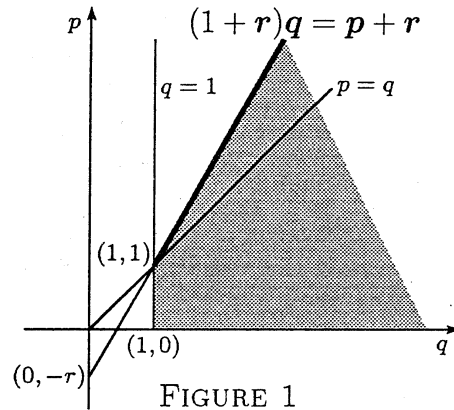


FIGURE 1

We remark that Theorem F yields the Löwner-Heinz theorem when we put $r = 0$. Alternative proofs of Theorem F are given in [6] and [20] and also an elementary one page proof in [10]. Tanahashi [22] showed that the domain drawn for p , q and r in the Figure 1 is the best possible one for Theorem F.

On the other hand, related to p -hyponormal, log-hyponormal and paranormal (i.e., $\|T^2x\| \geq \|Tx\|^2$ for every unit vector $x \in H$) operators, we introduced classes of operators in [15] as follows:

Definition 1 ([15]).

(i) *An operator T belongs to class A if $|T^2| \geq |T|^2$ where $|T| = (T^*T)^{\frac{1}{2}}$.*

(ii) *For each $k > 0$, an operator T belongs to class A(k) if*

$$(1.1) \quad (T^*|T|^{2k}T)^{\frac{1}{k+1}} \geq |T|^2.$$

(iii) *For each $k > 0$, an operator T is said to be absolute- k -paranormal if*

$$\| |T|^k Tx \| \geq \|Tx\|^{k+1}$$

holds for every unit vector $x \in H$.

We remark that class $A(1)$ equals class A , and absolute-1-paranormal equals paranormal. Related to these classes, we obtained the following result on inclusion relations among them in [15].

Theorem B ([15]).

- (i) Every log-hyponormal operator is a class $A(k)$ operator for all $k > 0$.
- (ii) Every invertible class $A(k)$ operator is a class $A(l)$ operator for $l \geq k > 0$.
- (iii) Every absolute- k -paranormal operator is a absolute- l -paranormal operator for $l \geq k > 0$.
- (iv) For each $k > 0$, every class $A(k)$ operator is a absolute- k -paranormal operator.

Inclusion relations among the classes of operators mentioned above can be expressed as follows:

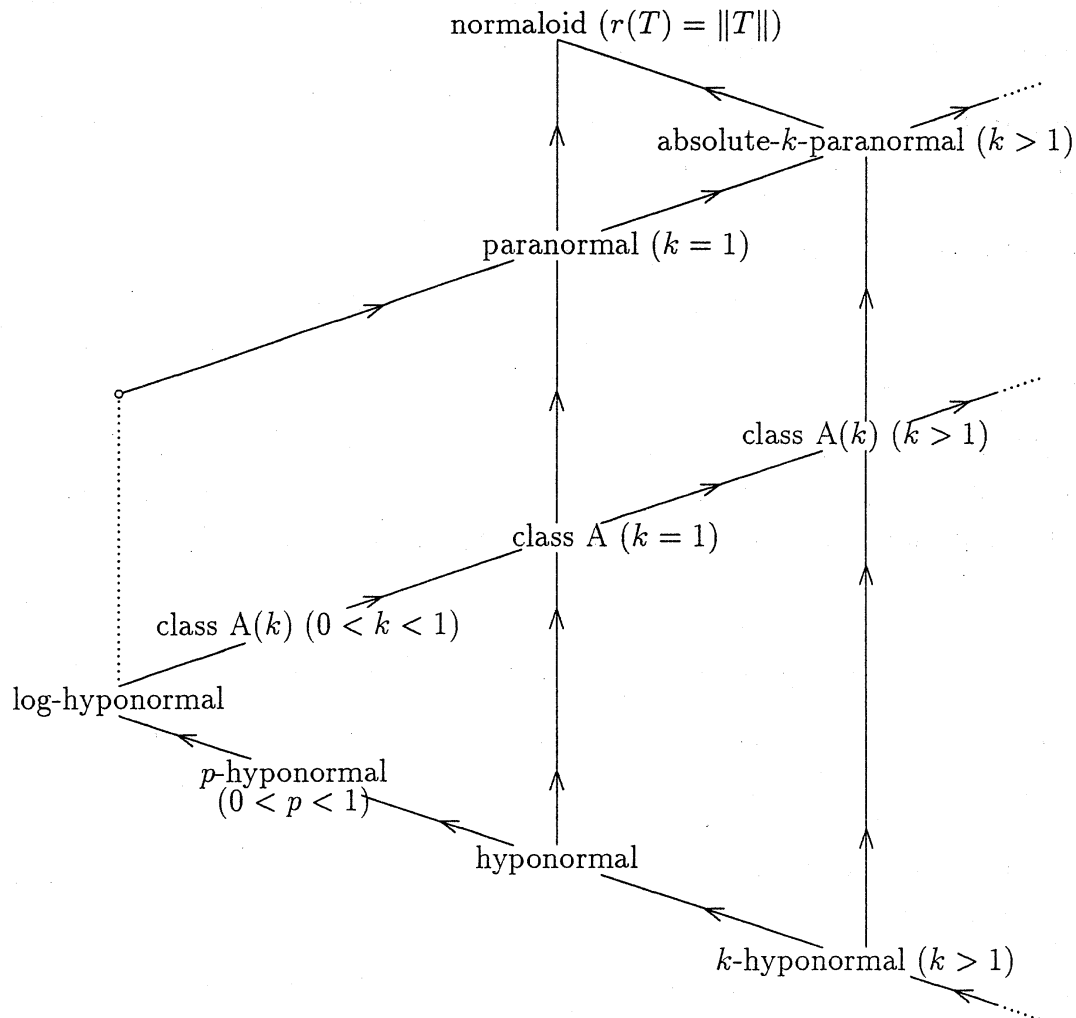


FIGURE 2

Related to Theorem A.1 and Theorem A.2 on powers of p -hyponormal and log-hyponormal operators, Ito [18] showed the following results on powers of class A operators as follows:

Theorem C.1 ([18]). *If T is an invertible class A operator, then T^n is also a class A operator for all positive integer n .*

Theorem C.2 ([18]). *Let T be an invertible class A operator. Then*

$$(i) \quad |T^n|^{\frac{2}{n}} \geq \cdots \geq |T^2| \geq |T|^2$$

and

$$(ii) \quad |T^*|^2 \geq |T^{2*}| \geq \cdots \geq |T^{n*}|^{\frac{2}{n}}$$

hold for all positive integer n .

It is interesting to point out that these theorems are parallel results to the following theorems on paranormal operators.

Theorem C.3 ([8]). *If T is a paranormal operator, then T^n is also a paranormal operator for all positive integer n .*

Theorem C.4 ([8, 18]). *Let T be a paranormal operator. Then*

$$\|T^n x\|^{\frac{2}{n}} \geq \cdots \geq \|T^2 x\| \geq \|Tx\|^2$$

hold for every unit vector $x \in H$ and all positive integer n

In this report, firstly, we shall show a result on powers of invertible class $A(k)$ operators for $k \in (0, 1]$ in Theorem 1, which is more precise result than Theorem C.1. Secondly, we shall show similar results to Theorem 1 for related classes to class $A(k)$.

2. POWERS OF CLASS $A(k)$ OPERATORS

Theorem 1. *If T is an invertible class $A(k)$ operator for $k \in (0, 1]$, then T^n is a class $A(\frac{k}{n})$ operator for all positive integer n .*

Corollary 2. *If T is an invertible class A operator, then T^n is a class $A(\frac{1}{n})$ operator for all positive integer n .*

By using (ii) of Theorem B, Corollary 2 yields Theorem C.1 since class $A(\frac{1}{n})$ is included in class A, so that Corollary 2 is a more precise result than Theorem C.1.

To prove Theorem 1, we prepare the following Proposition 3 and Lemma F.

Proposition 3. T is a class $A(k)$ operator for $k > 0$ if and only if

$$(2.1) \quad (|T^*||T|^{2k}|T^*|)^{\frac{1}{k+1}} \geq |T^*|^2.$$

Lemma F ([11, 15]). Let A and B be invertible operators. Then

$$(BAA^*B^*)^\lambda = BA(A^*B^*BA)^{\lambda-1}A^*B^*$$

holds for any real number λ .

Proof of Proposition 3. Let $T = U|T|$ be the polar decomposition of T . Then $T^* = U^*|T^*|$ is also the polar decomposition of T^* . Suppose that T is a class $A(k)$ operator. Then

$$\begin{aligned} (|T^*||T|^{2k}|T^*|)^{\frac{1}{k+1}} &= UU^*(|T^*||T|^{2k}|T^*|)^{\frac{1}{k+1}}UU^* \\ &= U(U^*|T^*||T|^{2k}|T^*|U)^{\frac{1}{k+1}}U^* \\ &= U(T^*|T|^{2k}T)^{\frac{1}{k+1}}U^* \\ &\geq U|T|^2U^* \quad \text{since } T \text{ is a class } A(k) \text{ operator} \\ &= |T^*|^2. \end{aligned}$$

Hence (2.1) holds.

Conversely, suppose that (2.1) holds. Then

$$\begin{aligned} (T^*|T|^{2k}T)^{\frac{1}{k+1}} &= (U^*|T^*||T|^{2k}|T^*|U)^{\frac{1}{k+1}} \\ &= U^*(|T^*||T|^{2k}|T^*|)^{\frac{1}{k+1}}U \\ &\geq U^*|T^*|^2U \quad \text{by (2.1)} \\ &= |T|^2. \end{aligned}$$

Hence T is a class $A(k)$ operator.

Whence the proof of Proposition 3 is complete. \square

Proof of Theorem 1. Suppose that T is an invertible class $A(k)$ operator for $k \in (0, 1]$. Then we have

$$(2.1) \quad (|T^*||T|^{2k}|T^*|)^{\frac{1}{k+1}} \geq |T^*|^2$$

by Proposition 3. By (ii) of Theorem B, T is a class A operator, and also we have the following inequalities by Theorem C.2:

$$(2.2) \quad |T^n|^{\frac{2}{n}} \geq |T|^2,$$

$$(2.3) \quad |T^*|^2 \geq |T^{n*}|^{\frac{2}{n}}.$$

Then we have

$$(2.4) \quad (|T^*||T^n|^{\frac{2k}{n}}|T^*|)^{\frac{1}{k+1}} \geq (|T^*||T|^{2k}|T^*|)^{\frac{1}{k+1}} \geq |T^*|^2$$

by (2.1), (2.2) and Löwner-Heinz theorem. (2.4) implies the following (2.5) by Lemma F:

$$(2.5) \quad |T^*| |T^n|^{\frac{k}{n}} (|T^n|^{\frac{k}{n}} |T^*|^2 |T^n|^{\frac{k}{n}})^{\frac{1}{k+1}-1} |T^n|^{\frac{k}{n}} |T^*| \geq |T^*|^2.$$

(2.5) is equivalent to

$$(2.6) \quad |T^n|^{\frac{2k}{n}} \geq (|T^n|^{\frac{k}{n}} |T^*|^2 |T^n|^{\frac{k}{n}})^{\frac{k}{k+1}}.$$

By (2.3), (2.6) and Löwner-Heinz theorem, we have

$$(2.7) \quad |T^n|^{\frac{2k}{n}} \geq (|T^n|^{\frac{k}{n}} |T^*|^2 |T^n|^{\frac{k}{n}})^{\frac{k}{k+1}} \geq (|T^n|^{\frac{k}{n}} |T^{n*}|^{\frac{2}{n}} |T^n|^{\frac{k}{n}})^{\frac{k}{k+1}}.$$

(2.7) implies the following by Lemma F:

$$|T^n|^{\frac{2k}{n}} \geq |T^n|^{\frac{k}{n}} |T^{n*}|^{\frac{1}{n}} (|T^{n*}|^{\frac{1}{n}} |T^n|^{\frac{2k}{n}} |T^{n*}|^{\frac{1}{n}})^{\frac{k}{k+1}-1} |T^{n*}|^{\frac{1}{n}} |T^n|^{\frac{k}{n}}.$$

Then we have

$$(|T^{n*}|^{\frac{1}{n}} |T^n|^{\frac{2k}{n}} |T^{n*}|^{\frac{1}{n}})^{\frac{1}{k+1}} \geq |T^{n*}|^{\frac{2}{n}}.$$

Put $A = (|T^{n*}|^{\frac{1}{n}} |T^n|^{\frac{2k}{n}} |T^{n*}|^{\frac{1}{n}})^{\frac{1}{k+1}}$ and $B = |T^{n*}|^{\frac{2}{n}}$, then $A \geq B > 0$. By using (i) of Theorem F, we have

$$(2.8) \quad (B^{\frac{r}{2}} A^p B^{\frac{r}{2}})^{\frac{1+r}{p+r}} \geq B^{1+r} \quad \text{for } p \geq 1 \text{ and } r \geq 0.$$

Put $p = k + 1 \geq 1$ and $r = n - 1 \geq 0$ in (2.8). Then we have

$$(2.9) \quad (B^{\frac{n-1}{2}} A^{k+1} B^{\frac{n-1}{2}})^{\frac{n}{k+n}} \geq B^n.$$

(2.9) is equivalent to

$$\left\{ |T^{n*}|^{\frac{n-1}{n}} (|T^{n*}|^{\frac{1}{n}} |T^n|^{\frac{2k}{n}} |T^{n*}|^{\frac{1}{n}})^{\frac{k+1}{k+1}} |T^{n*}|^{\frac{n-1}{n}} \right\}^{\frac{n}{k+n}} \geq |T^{n*}|^2.$$

Then we have

$$(2.10) \quad (|T^{n*}| |T^n|^{\frac{2k}{n}} |T^{n*}|)^{\frac{1}{k+1}} \geq |T^{n*}|^2.$$

Hence T^n is a class $A(\frac{k}{n})$ operator by Proposition 3. □

Proof of Corollary 2. Put $k = 1$ in Theorem 1. □

3. POWERS OF CLASS $AI(s, t)$ OPERATORS

Very recently, the following classes of operators were defined in [7] as extensions of class $A(k)$.

Definition 2 ([7]).

(i) For each $s > 0$ and $t > 0$, an operator T belongs to class $A(s, t)$ if

$$(3.1) \quad (|T^*|^t |T|^{2s} |T^*|^t)^{\frac{t}{s+t}} \geq |T^*|^{2t}.$$

(ii) For each $s > 0$ and $t > 0$, an operator T belongs to class $AI(s, t)$ if T is an invertible class $A(s, t)$ operator.

We remark that class $A(k)$ coincides with class $A(k, 1)$ by Proposition 3. Related to class $A(s, t)$ operators, the following theorem was obtained in [7] as a nice application of Theorem F.

Theorem D ([7]).

- (i) For each $s > 0$ and $t > 0$, every class $A(s, t)$ operator is a class $A(s, r)$ operator for $r \geq t > 0$.
- (ii) For each $s > 0$ and $t > 0$, every class $AI(s, t)$ operator is a class $AI(p, r)$ operator for $p \geq s > 0$ and $r \geq t > 0$.

On the other hand, Aluthge and Wang defined w -hyponormal operators in [4] which was related to hyponormal operators as follows: An operator T is said to be w -hyponormal if $|\tilde{T}| \geq |T| \geq |(\tilde{T})^*|$ where $\tilde{T} = |T|^{\frac{1}{2}} U |T|^{\frac{1}{2}}$ and $T = U |T|$ is the polar decomposition of T . $\tilde{T} = |T|^{\frac{1}{2}} U |T|^{\frac{1}{2}}$ is called Aluthge transformation of T . Aluthge transformation was studied in [1, 2, 12, 13, 17, 21, 24]. Related to w -hyponormal operators, the following result was shown in [5].

Theorem E.1 ([5]). An operator T is a w -hyponormal operator if and only if

$$(|T^*|^{\frac{1}{2}} |T| |T^*|^{\frac{1}{2}})^{\frac{1}{2}} \geq |T^*| \quad \text{and} \quad |T| \geq (|T|^{\frac{1}{2}} |T^*| |T|^{\frac{1}{2}})^{\frac{1}{2}}.$$

By Theorem E.1, an invertible w -hyponormal operator coincides with a class $AI(\frac{1}{2}, \frac{1}{2})$ operator since the first inequality in Theorem E.1 is equivalent to the second inequality in Theorem E.1 in case T is invertible by Lemma F. Moreover, the following Theorem E.2 for w -hyponormal operators was shown by Aluthge and Wang [5].

Theorem E.2 ([5]). If T is an invertible w -hyponormal operator, then T^2 is also a w -hyponormal operator.

In this section, we shall show the following results for class $AI(s, t)$ operators and w -hyponormal operators as parallel results to Theorem 1 for class $A(k)$ operators.

Theorem 4. If T is a class $AI(s, t)$ operator for $s, t \in (0, 1]$, then T^n is a class $AI(\frac{s}{n}, \frac{t}{n})$ operator for all positive integer n .

Corollary 5. *If T is an invertible w -hyponormal operator, then T^n is a class $\text{AI}(\frac{1}{2n}, \frac{1}{2n})$ operator for all positive integer n .*

By (ii) of Theorem D and Theorem E.1, Corollary 5 yields Theorem E.2 since class $\text{AI}(\frac{1}{2n}, \frac{1}{2n})$ is included in w -hyponormal for all positive integer n .

Proof of Theorem 4. Suppose that T is a class $\text{AI}(s, t)$ operator for $s, t \in (0, 1]$, i.e.,

$$(3.1) \quad (|T^*|^t |T|^{2s} |T^*|^t)^{\frac{t}{s+t}} \geq |T^*|^{2t}.$$

By (ii) of Theorem D and Proposition 3, T is a class A operator, and also we have the following inequalities by Theorem C.2:

$$(2.2) \quad |T^n|^{\frac{2}{n}} \geq |T|^2,$$

$$(2.3) \quad |T^*|^2 \geq |T^{n*}|^{\frac{2}{n}}.$$

(2.2) and (2.3) imply the following (3.2) and (3.3), respectively, by Löwner-Heinz theorem for $s \in (0, 1]$ and $t \in (0, 1]$:

$$(3.2) \quad |T^n|^{\frac{2s}{n}} \leq |T|^{2s},$$

$$(3.3) \quad |T^*|^{2t} \geq |T^{n*}|^{\frac{2t}{n}}.$$

Then we have

$$(3.4) \quad (|T^*|^t |T^n|^{\frac{2s}{n}} |T^*|^t)^{\frac{t}{s+t}} \geq (|T^*|^t |T|^{2s} |T^*|^t)^{\frac{t}{s+t}} \geq |T^*|^{2t}$$

by (3.1), (3.2) and Löwner-Heinz theorem. (3.4) implies the following (3.5) by Lemma F:

$$(3.5) \quad |T^*|^t |T^n|^{\frac{s}{n}} (|T^n|^{\frac{s}{n}} |T^*|^{2t} |T^n|^{\frac{s}{n}})^{\frac{t}{s+t}-1} |T^n|^{\frac{s}{n}} |T^*|^t \geq |T^*|^{2t}.$$

(3.5) is equivalent to

$$(3.6) \quad |T^n|^{\frac{2s}{n}} \geq (|T^n|^{\frac{s}{n}} |T^*|^{2t} |T^n|^{\frac{s}{n}})^{\frac{s}{s+t}}.$$

By (3.3), (3.6) and Löwner-Heinz theorem, we have

$$(3.7) \quad |T^n|^{\frac{2s}{n}} \geq (|T^n|^{\frac{s}{n}} |T^*|^{2t} |T^n|^{\frac{s}{n}})^{\frac{s}{s+t}} \geq (|T^n|^{\frac{s}{n}} |T^{n*}|^{\frac{2t}{n}} |T^n|^{\frac{s}{n}})^{\frac{s}{s+t}}.$$

(3.7) implies the following by Lemma F:

$$|T^n|^{\frac{2s}{n}} \geq |T^n|^{\frac{s}{n}} |T^{n*}|^{\frac{t}{n}} (|T^{n*}|^{\frac{t}{n}} |T^n|^{\frac{2s}{n}} |T^{n*}|^{\frac{t}{n}})^{\frac{s}{s+t}-1} |T^{n*}|^{\frac{t}{n}} |T^n|^{\frac{s}{n}}.$$

Then we have

$$(|T^{n*}|^{\frac{t}{n}} |T^n|^{\frac{2s}{n}} |T^{n*}|^{\frac{t}{n}})^{\frac{\frac{t}{n}}{\frac{s}{n}+\frac{t}{n}}} \geq |T^{n*}|^{\frac{2t}{n}}.$$

Hence T^n is a class $\text{AI}(\frac{s}{n}, \frac{t}{n})$ operator. □

Proof of Corollary 5. Put $s = \frac{1}{2}$ and $t = \frac{1}{2}$ in Theorem 4. Then we obtain Corollary 5 since the class of all invertible w -hyponormal operators equals class $AI(\frac{1}{2}, \frac{1}{2})$. \square

4. CONCLUDING REMARKS

Firstly, it is interesting to point out the contrast between the following two facts: Theorem A.1 asserts that if T is a p -hyponormal operator for $p \in (0, 1]$, then T^n belongs to the class of $(\frac{p}{n})$ -hyponormal operators which is a larger class of operators than the class of p -hyponormal operators to which T belongs. Contrary to Theorem A.1, Theorem 1 asserts that if T is an invertible class $A(k)$ operator for $k \in (0, 1]$, then T^n belongs to class $A(\frac{k}{n})$ which is a smaller class of operators than class $A(k)$ to which T belongs.

Secondly, it is shown in (i) of Theorem B that every log-hyponormal operator is a class $A(k)$ operator for all $k > 0$. Here, we shall discuss a more precise relation between class $A(k)$ and the class of log-hyponormal operators than (i) of Theorem B. Assume that T is an invertible class $A(k)$ operator. Then we have

$$(2.1) \quad (|T^*||T|^{2k}|T^*|)^{\frac{1}{k+1}} \geq |T^*|^2$$

by Proposition 3. Then by using Lemma F, (2.1) is equivalent to the following (4.1):

$$(4.1) \quad |T^*||T|^k(|T|^k|T^*|^2|T|^k)^{\frac{-k}{k+1}}|T|^k|T^*| \geq |T^*|^2.$$

Hence an invertible class $A(k)$ operator satisfies the following inequality by (4.1):

$$|T|^{2k} \geq (|T|^k|T^*|^2|T|^k)^{\frac{k}{k+1}}.$$

Then

$$(4.2) \quad \log |T|^{2k} \geq \log(|T|^k|T^*|^2|T|^k)^{\frac{k}{k+1}}$$

holds since $\log t$ is an operator monotone function. (4.2) is equivalent to

$$(4.3) \quad \log |T|^{2(k+1)} \geq \log(|T|^k|T^*|^2|T|^k).$$

Let $k \rightarrow +0$ in (4.3). Then we have $\log T^*T \geq \log TT^*$. Briefly speaking, the class of log-hyponormal operators can be regarded as invertible class $A(0)$. And it is well known that log-hyponormal also can be regarded as 0-hyponormal. It is interesting to point out this contrast.

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